Study of the effect of parameters on the decay rate of a fourth order problem

Annual Meeting of the COMPLEX Doctoral School 2023

Damien Galant

CERAMATHS/DMATHS
Université Polytechnique Hauts-de-France

Département de Mathématique
Université de Mons
F.R.S.-FNRS Research Fellow

Joint work with Christophe Troestler (UMONS) and Virginie Régnier (CERAMATHS/DMATHS, UPHF)

Monday 13 November 2023

The equation

We study the evolution equation

$$\partial_{tt} u(x,t) + a\partial_{xxxx} u(x,t) + b\partial_t u(x,t) + \alpha\partial_t u(\xi,t)\delta_\xi = 0,$$

where $(x,t) \in (0,1) \times (0,+\infty)$.

The equation

We study the evolution equation

$$\partial_{tt} u(x,t) + a\partial_{xxxx} u(x,t) + b\partial_t u(x,t) + \alpha\partial_t u(\xi,t)\delta_\xi = 0,$$

where $(x,t) \in (0,1) \times (0,+\infty)$. Here,

- a, b: fixed positive constants;

The equation

We study the evolution equation

$$\partial_{tt} u(x,t) + a\partial_{xxxx} u(x,t) + b\partial_t u(x,t) + \alpha \partial_t u(\xi,t)\delta_\xi = 0,$$

where $(x,t) \in (0,1) \times (0,+\infty)$. Here,

- a, b: fixed positive constants;
- α: positive constant, δ_ξ: Dirac mass at the point $\xi \in (0,1)$.

The equation

We study the evolution equation

$$\partial_{tt} u(x,t) + a\partial_{xxxx} u(x,t) + b\partial_t u(x,t) + \alpha\partial_t u(\xi,t)\delta_\xi = 0,$$

where $(x,t) \in (0,1) \times (0,+\infty)$. Here,

- a, b: fixed positive constants;
- α: positive constant, δ_ξ: Dirac mass at the point $\xi \in (0,1)$.

This problem is motivated by the study of vibration in bridges. Then:

The equation

We study the evolution equation

$$\partial_{tt} u(x,t) + a\partial_{xxxx} u(x,t) + b\partial_t u(x,t) + \alpha \partial_t u(\xi,t)\delta_\xi = 0,$$

where $(x,t) \in (0,1) \times (0,+\infty)$. Here,

- a, b: fixed positive constants;
- α: positive constant, δ_ξ: Dirac mass at the point $\xi \in (0,1)$.

This problem is motivated by the study of vibration in bridges. Then:

- u: transverse displacement of the bridge deck (identified with $[0,1]$);

The equation

We study the evolution equation

$$\partial_{tt}u(x,t) + a\partial_{xxxx}u(x,t) + b\partial_t u(x,t) + \alpha\partial_t u(\xi,t)\delta_\xi = 0,$$

where $(x,t) \in (0,1) \times (0,+\infty)$. Here,

- a, b: fixed positive constants;
- α: positive constant, δ_ξ: Dirac mass at the point $\xi \in (0,1)$.

This problem is motivated by the study of vibration in bridges. Then:

- u: transverse displacement of the bridge deck (identified with $[0,1]$);
- δ_ξ: presence of a shape memory alloy cable at $x = \xi$.

Boundary and initial conditions

We couple the equations with boundary conditions

$$u(0,t) = u(1,t) = \partial_x^2 u(0,t) = \partial_x^2 u(1,t) = 0, \qquad t \in (0,+\infty),$$

Boundary and initial conditions

We couple the equations with boundary conditions

$$u(0,t) = u(1,t) = \partial_x^2 u(0,t) = \partial_x^2 u(1,t) = 0, \qquad t \in (0,+\infty),$$

and initial conditions

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \qquad x \in (0,1).$$

Energy and dissipation

We define the energy of a solution $u(x,t)$ by

$$E(t) := \frac{1}{2}\int_0^1 \left(|\partial_t u|^2 + a|\partial_{xx} u|^2\right) \mathrm{d}x.$$

Energy and dissipation

We define the energy of a solution $u(x,t)$ by

$$E(t) := \frac{1}{2}\int_0^1 \left(|\partial_t u|^2 + a|\partial_{xx} u|^2\right) \mathrm{d}x.$$

Differentiating (formally) and integrating by parts, we obtain

$$\partial_t E(t) = -b\left(\int_0^1 |\partial_t u|^2 \,\mathrm{d}x\right) - \alpha|\partial_t u(\xi, t)|^2$$

Energy and dissipation

We define the energy of a solution $u(x,t)$ by

$$E(t) := \frac{1}{2}\int_0^1 \left(|\partial_t u|^2 + a|\partial_{xx} u|^2 \right) \mathrm{d}x.$$

Differentiating (formally) and integrating by parts, we obtain

$$\partial_t E(t) = -b\left(\int_0^1 |\partial_t u|^2 \,\mathrm{d}x\right) - \alpha|\partial_t u(\xi,t)|^2 \le 0.$$

Energy and dissipation

We define the energy of a solution $u(x,t)$ by

$$E(t) := \frac{1}{2}\int_0^1 \left(|\partial_t u|^2 + a|\partial_{xx} u|^2\right) \mathrm{d}x.$$

Differentiating (formally) and integrating by parts, we obtain

$$\partial_t E(t) = -b\left(\int_0^1 |\partial_t u|^2 \,\mathrm{d}x\right) - \alpha|\partial_t u(\xi,t)|^2 \leq 0.$$

Hence, the system is dissipative, in the sense that the energy decreases.

The damping rate

Using semigroup theory in a suitable functional setting, one can show that the evolution problem is well-posed.

The damping rate

Using semigroup theory in a suitable functional setting, one can show that the evolution problem is well-posed.

Theorem (Régnier (2022))

The optimal energy decay rate of the equation, i.e. the smallest $\omega(\alpha) < 0$ *such that, for all initial conditions* U_0*, there exists* $C(U_0) > 0$ *with*

$$E(t) \leq C(U_0)e^{2\omega(\alpha)t}$$

for all $t \geq 0$ *is given by*

$$\omega(\alpha) := \sup\left\{\Re(\mu) \mid \mu \textit{ is an eigenvalue of the problem}\right\}.$$

The role of the parameter α

A natural question

How does the decay rate $\omega(\alpha)$ depend on α?

The role of the parameter α

A natural question

How does the decay rate $\omega(\alpha)$ depend on α?

An important assumption

To avoid any “resonance phenomena”, we will assume that $\xi \notin \mathbb{Q}$.

The characteristic equation

Proposition

A complex number $\mu \in \mathbb{C} \setminus \{-b, 0\}$ *is an eigenvalue of the problem if and only if it satisfies the characteristic equation*

The characteristic equation

Proposition

A complex number $\mu \in \mathbb{C} \setminus \{-b, 0\}$ *is an eigenvalue of the problem if and only if it satisfies the characteristic equation*

$$(\mu + b)\sinh(\lambda)\sin(\lambda)$$
$$+\alpha\lambda\Big[\sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi)) - \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi))\Big] = 0,$$

The characteristic equation

Proposition

A complex number $\mu \in \mathbb{C} \setminus \{-b, 0\}$ *is an eigenvalue of the problem if and only if it satisfies the characteristic equation*

$$\begin{aligned} &(\mu + b)\sinh(\lambda)\sin(\lambda) \\ &+ \alpha\lambda\Big[\sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi)) - \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi))\Big] = 0, \end{aligned}$$

where

$$\lambda(\mu) := \sqrt[4]{-\frac{b\mu + \mu^2}{a}}.$$

The characteristic equation

Proposition

A complex number $\mu \in \mathbb{C} \setminus \{-b, 0\}$ is an eigenvalue of the problem if and only if it satisfies the characteristic equation

$$(\mu + b)\sinh(\lambda)\sin(\lambda) + \alpha\lambda\Big[\sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi)) - \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi))\Big] = 0,$$

where

$$\lambda(\mu) := \sqrt[4]{-\frac{b\mu + \mu^2}{a}}.$$

Remark

Replacing λ by $i\lambda$, $-\lambda$ or $-i\lambda$ leads to an equivalent equation.

The characteristic equation

Finding the eigenvalues of the problem amounts to find roots of the function

$$F(\mu; \alpha) := 2\mu(\mu + b)F_0(\lambda) + \alpha\mu\lambda F_1(\lambda),$$

The characteristic equation

Finding the eigenvalues of the problem amounts to find roots of the function

$$F(\mu;\alpha) := 2\mu(\mu+b)F_0(\lambda) + \alpha\mu\lambda F_1(\lambda),$$

where

$$\lambda(\mu) := \sqrt[4]{-\frac{b\mu+\mu^2}{a}},$$

The characteristic equation

Finding the eigenvalues of the problem amounts to find roots of the function

$$F(\mu;\alpha) := 2\mu(\mu+b)F_0(\lambda) + \alpha\mu\lambda F_1(\lambda),$$

where

$$\lambda(\mu) := \sqrt[4]{-\frac{b\mu+\mu^2}{a}},$$

$$F_0(\lambda) := \sinh(\lambda)\sin(\lambda),$$

The characteristic equation

Finding the eigenvalues of the problem amounts to find roots of the function

$$F(\mu;\alpha) := 2\mu(\mu+b)F_0(\lambda) + \alpha\mu\lambda F_1(\lambda),$$

where

$$\lambda(\mu) := \sqrt[4]{-\frac{b\mu+\mu^2}{a}},$$

$$F_0(\lambda) := \sinh(\lambda)\sin(\lambda),$$

$$F_1(\lambda) := \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi)) - \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)).$$

The characteristic equation

Finding the eigenvalues of the problem amounts to find roots of the function

$$F(\mu;\alpha) := 2\mu(\mu+b)F_0(\lambda) + \alpha\mu\lambda F_1(\lambda),$$

where

$$\lambda(\mu) := \sqrt[4]{-\frac{b\mu+\mu^2}{a}},$$
$$F_0(\lambda) := \sinh(\lambda)\sin(\lambda),$$
$$F_1(\lambda) := \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi)) - \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)).$$

Remark

- $\alpha = 0 \rightsquigarrow$ *roots of* F_0*;*

The characteristic equation

Finding the eigenvalues of the problem amounts to find roots of the function

$$F(\mu;\alpha) := 2\mu(\mu+b)F_0(\lambda) + \alpha\mu\lambda F_1(\lambda),$$

where

$$\lambda(\mu) := \sqrt[4]{-\frac{b\mu+\mu^2}{a}},$$
$$F_0(\lambda) := \sinh(\lambda)\sin(\lambda),$$
$$F_1(\lambda) := \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi)) - \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)).$$

Remark

- $\alpha = 0 \rightsquigarrow$ *roots of* F_0*;*
- $\alpha \to +\infty \rightsquigarrow$ *roots of* F_1*.*

Dependence of the roots on parameters

A general fact from complex analysis

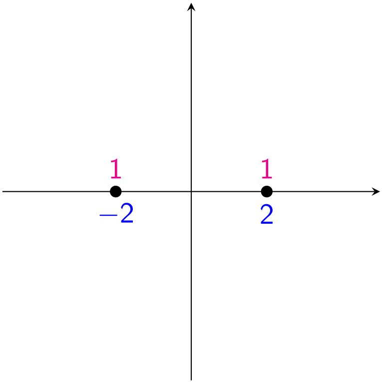
Theorem ("Holomorphic implicit function Theorem", very roughly stated)

Roots of holomorphic functions depend ***continuously, including multiplicities,*** *on the parameters, and the branches of roots are holomorphic.*

Dependence of the roots on parameters

A simple example: roots of $z \mapsto z^2 + c$

$$z \mapsto z^2 - 4$$

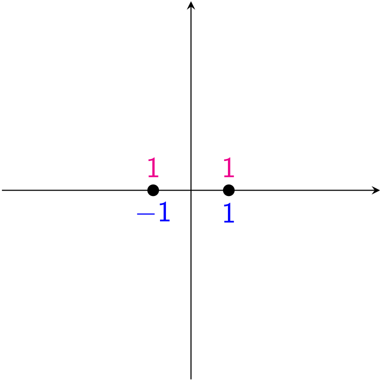


Blue: values. Red: multiplicities.

Dependence of the roots on parameters

A simple example: roots of $z \mapsto z^2 + c$

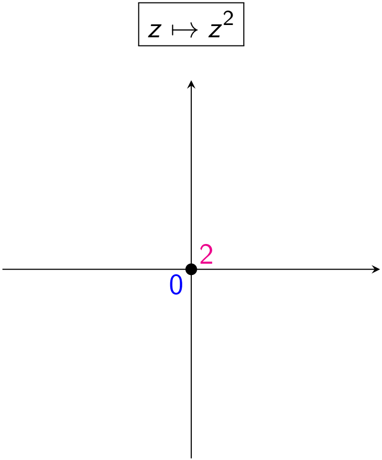
$$z \mapsto z^2 - 1$$



Blue: values. Red: multiplicities.

Dependence of the roots on parameters

A simple example: roots of $z \mapsto z^2 + c$

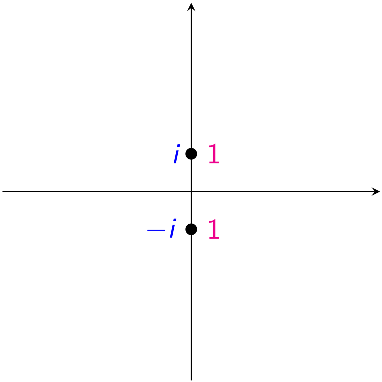


Blue: values. Red: multiplicities.

Dependence of the roots on parameters

A simple example: roots of $z \mapsto z^2 + c$

$$z \mapsto z^2 + 1$$

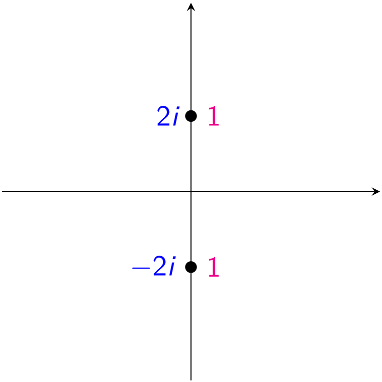


Blue: values. Red: multiplicities.

Dependence of the roots on parameters

A simple example: roots of $z \mapsto z^2 + c$

$$z \mapsto z^2 + 4$$



Blue: values. Red: multiplicities.

The case $\alpha = 0$: roots of $\lambda \mapsto F_0(\lambda)$

A computation

We recall that

$$F_0(\lambda) := \sinh(\lambda) \sin(\lambda).$$

The case $\alpha = 0$: roots of $\lambda \mapsto F_0(\lambda)$

A computation

We recall that

$$F_0(\lambda) := \sinh(\lambda)\sin(\lambda).$$

Therefore, the set of roots of F_0 is

$$\left\{k\pi, ik\pi \mid k \in \mathbb{Z}\right\},$$

The case $\alpha = 0$: roots of $\lambda \mapsto F_0(\lambda)$

A computation

We recall that

$$F_0(\lambda) := \sinh(\lambda)\sin(\lambda).$$

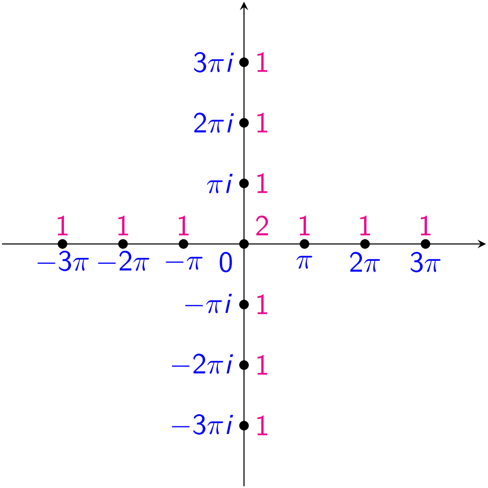
Therefore, the set of roots of F_0 is

$$\left\{k\pi, ik\pi \mid k \in \mathbb{Z}\right\},$$

and all have multiplicity one, except zero which has multiplicity two.

The case $\alpha = 0$: roots of $\lambda \mapsto F_0(\lambda)$

Graphical representation in the λ plane



The case $\alpha = 0$: roots of $\mu \mapsto F_0(\lambda(\mu))$

Graphical representation in the μ plane ($a = 0.05, b = 3$)

$$\lambda(\mu) = \sqrt[4]{-\frac{b\mu + \mu^2}{a}}$$

i.e.

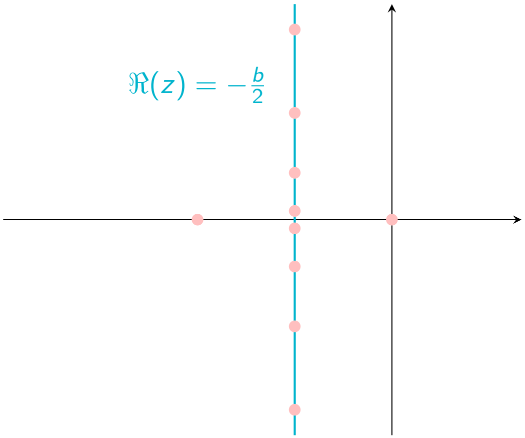
$$\mu^2 + b\mu + a\lambda^4 = 0$$

so that

$$\mu(\lambda) = \frac{-b \pm \sqrt{b^2 - 4a\lambda^4}}{2}$$

The case $\alpha = 0$: roots of $\mu \mapsto F_0(\lambda(\mu))$

Graphical representation in the μ plane ($a = 0.05$, $b = 3$)



$$\lambda(\mu) = \sqrt[4]{-\frac{b\mu + \mu^2}{a}}$$

i.e.

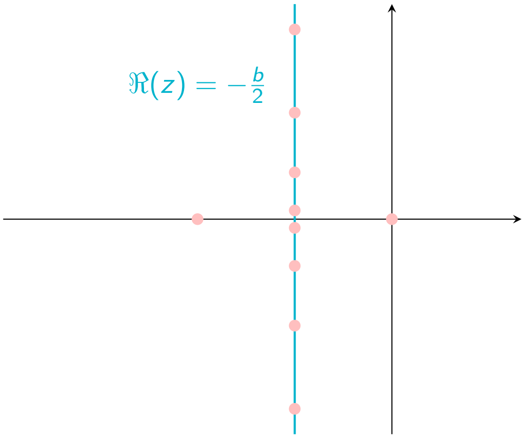
$$\mu^2 + b\mu + a\lambda^4 = 0$$

so that

$$\mu(\lambda) = \frac{-b \pm \sqrt{b^2 - 4a\lambda^4}}{2}$$

The case $\alpha = 0$: roots of $\mu \mapsto F_0(\lambda(\mu))$

Graphical representation in the μ plane ($a = 0.05$, $b = 3$)



$$\lambda(\mu) = \sqrt[4]{-\frac{b\mu + \mu^2}{a}}$$

i.e.

$$\mu^2 + b\mu + a\lambda^4 = 0$$

so that

$$\mu(\lambda) = \frac{-b \pm \sqrt{b^2 - 4a\lambda^4}}{2}$$

Note: 0 is a root, but is *not* an eigenvalue!

Roots of $\lambda \mapsto F_1(\lambda)$

The strategy: a continuation argument

We write

$$F_1(\lambda) = s(\lambda) - t(\lambda)$$

where

$$s(\lambda) := \sin(\lambda) \sinh(\lambda\xi) \sinh(\lambda(1-\xi))$$
$$t(\lambda) := \sinh(\lambda) \sin(\lambda\xi) \sin(\lambda(1-\xi)).$$

Roots of $\lambda \mapsto F_1(\lambda)$

The strategy: a continuation argument

We write

$$F_1(\lambda) = s(\lambda) - t(\lambda)$$

where

$$s(\lambda) := \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi))$$
$$t(\lambda) := \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)).$$

Strategy: study roots of

$$\tilde{F}_\gamma(\lambda) := s(\lambda) - \gamma t(\lambda).$$

as γ varies from 0 to 1.

Roots of $\lambda \mapsto F_1(\lambda)$

Roots of s and t: a computation

We have

$$s(\lambda) := \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi))$$
$$t(\lambda) := \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)),$$

Roots of $\lambda \mapsto F_1(\lambda)$

Roots of s and t: a computation

We have

$$s(\lambda) := \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi))$$
$$t(\lambda) := \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)),$$

so that

$$\left\{\lambda \in \mathbb{C} \;\middle|\; s(\lambda) = 0\right\} = \left\{k\pi, i\frac{k\pi}{\xi}, i\frac{k\pi}{1-\xi} \;\middle|\; k \in \mathbb{Z}\right\}$$

Roots of $\lambda \mapsto F_1(\lambda)$

Roots of s and t: a computation

We have

$$s(\lambda) := \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi))$$
$$t(\lambda) := \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)),$$

so that

$$\left\{\lambda \in \mathbb{C} \;\middle|\; s(\lambda) = 0\right\} = \left\{k\pi, i\frac{k\pi}{\xi}, i\frac{k\pi}{1-\xi} \;\middle|\; k \in \mathbb{Z}\right\}$$

and

$$\left\{\lambda \in \mathbb{C} \;\middle|\; t(\lambda) = 0\right\} = \left\{ik\pi, \frac{k\pi}{\xi}, \frac{k\pi}{1-\xi} \;\middle|\; k \in \mathbb{Z}\right\}.$$

Roots of $\lambda \mapsto F_1(\lambda)$

Roots of s and t: a computation

We have

$$s(\lambda) := \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi))$$
$$t(\lambda) := \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)),$$

so that

$$\left\{\lambda \in \mathbb{C} \,\middle|\, s(\lambda) = 0\right\} = \left\{k\pi, i\frac{k\pi}{\xi}, i\frac{k\pi}{1-\xi} \,\middle|\, k \in \mathbb{Z}\right\}$$

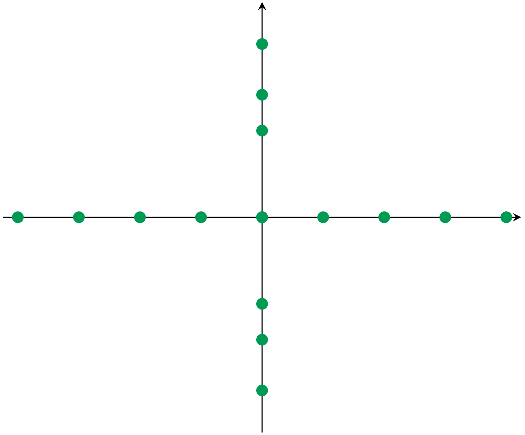
and

$$\left\{\lambda \in \mathbb{C} \,\middle|\, t(\lambda) = 0\right\} = \left\{ik\pi, \frac{k\pi}{\xi}, \frac{k\pi}{1-\xi} \,\middle|\, k \in \mathbb{Z}\right\}.$$

All those roots have multiplicity one, except 0.

Roots of $\lambda \mapsto F_1(\lambda)$

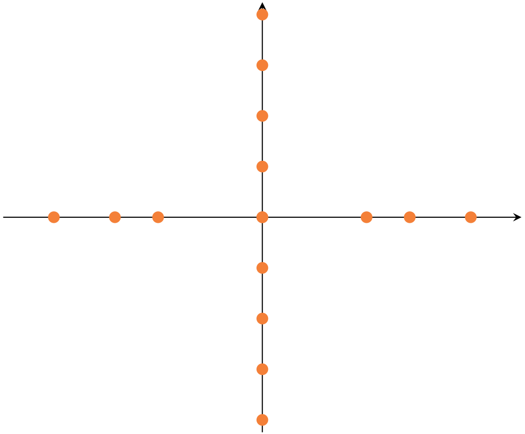
Roots of s and t: graphical representation in the λ plane



Green: roots of s

Roots of $\lambda \mapsto F_1(\lambda)$

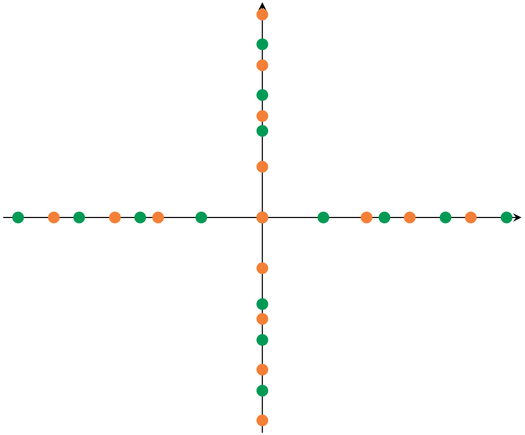
Roots of s and t: graphical representation in the λ plane



Orange: roots of t

Roots of $\lambda \mapsto F_1(\lambda)$

Roots of s and t: graphical representation in the λ plane

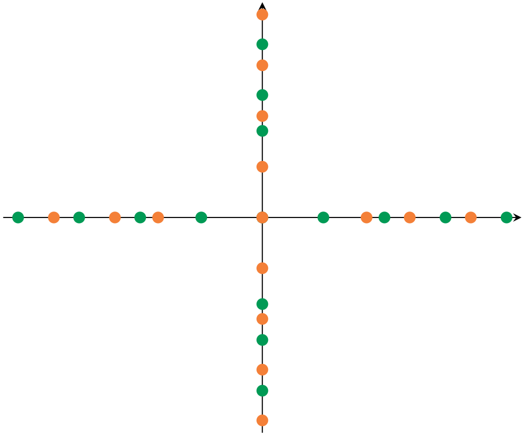


Green: roots of s

Orange: roots of t

Roots of $\lambda \mapsto F_1(\lambda)$

Roots of s and t: graphical representation in the λ plane



Green: roots of s
Orange: roots of t

Can you see something?

Roots of $\lambda \mapsto F_1(\lambda)$

A detour through number theory: Beatty's Theorem

Theorem (Rayleigh (1894) - Beatty (1927))

Let $0 < r < 1$ *be irrational. Define the sets*

$$A := \left\{ \left\lfloor \frac{n}{r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}, \qquad B := \left\{ \left\lfloor \frac{n}{1-r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}.$$

Roots of $\lambda \mapsto F_1(\lambda)$

A detour through number theory: Beatty's Theorem

Theorem (Rayleigh (1894) - Beatty (1927))

Let $0 < r < 1$ *be irrational. Define the sets*

$$A := \left\{ \left\lfloor \frac{n}{r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}, \qquad B := \left\{ \left\lfloor \frac{n}{1-r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}.$$

Then,

$$A \cap B = \emptyset, \qquad A \cup B = \mathbb{Z}^{>0}.$$

Roots of $\lambda \mapsto F_1(\lambda)$

A detour through number theory: Beatty's Theorem

Theorem (Rayleigh (1894) - Beatty (1927))

Let $0 < r < 1$ *be irrational. Define the sets*

$$A := \left\{ \left\lfloor \frac{n}{r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}, \qquad B := \left\{ \left\lfloor \frac{n}{1-r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}.$$

Then,

$$A \cap B = \emptyset, \qquad A \cup B = \mathbb{Z}^{>0}.$$

J. W. Strutt, 3rd Baron Rayleigh. The Theory of Sound. Vol. 1 (Second ed.). Macmillan (1894). p. 123.

Roots of $\lambda \mapsto F_1(\lambda)$

A detour through number theory: Beatty's Theorem

Theorem (Rayleigh (1894) - Beatty (1927))

Let $0 < r < 1$ *be irrational. Define the sets*

$$A := \left\{ \left\lfloor \frac{n}{r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}, \qquad B := \left\{ \left\lfloor \frac{n}{1-r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}.$$

Then,

$$A \cap B = \emptyset, \qquad A \cup B = \mathbb{Z}^{>0}.$$

J. W. Strutt, 3rd Baron Rayleigh. The Theory of Sound. Vol. 1 (Second ed.). Macmillan (1894). p. 123.

S. Beatty "Problem 3173". American Mathematical Monthly. 33 (3): p. 159 (1926).

Roots of $\lambda \mapsto F_1(\lambda)$

Beatty's Theorem: a numerical example

Let us take $r = \sqrt{2} - 1$. Then (using a little script),

$$A = \left\{ \left\lfloor \frac{n}{r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}$$

$$= \left\{ 2, 4, 7, 9, 12, 14, 16, 19, \dots \right\}$$

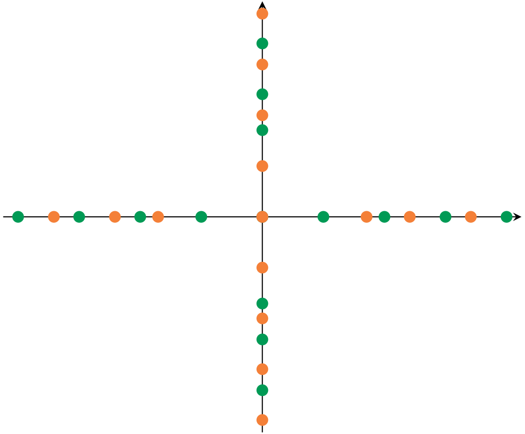
and

$$B = \left\{ \left\lfloor \frac{n}{1-r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}$$

$$= \left\{ 1, 3, 5, 6, 8, 10, 11, 13, 15, 17, 18, 20, \dots \right\}.$$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane

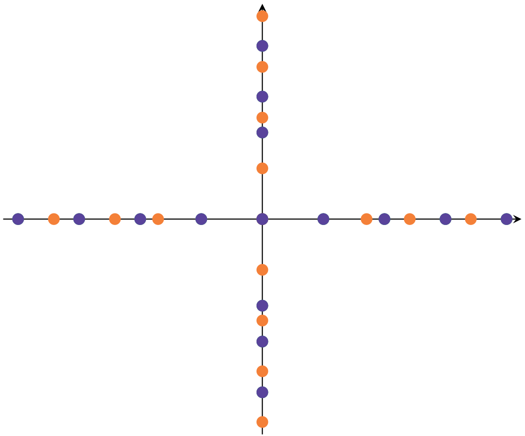


Green: roots of s

Orange: roots of t

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



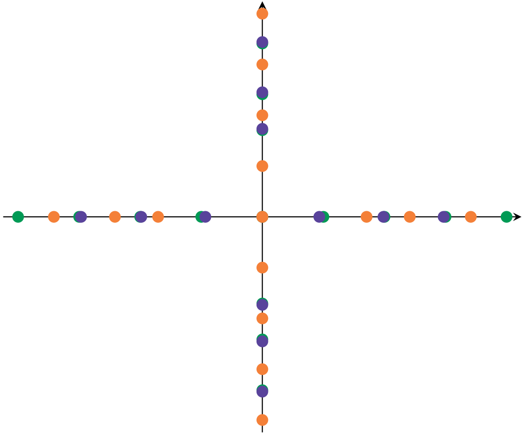
Green: roots of s

Orange: roots of t

$\gamma = 0$

Roots of $\lambda \mapsto F_1(\lambda)$

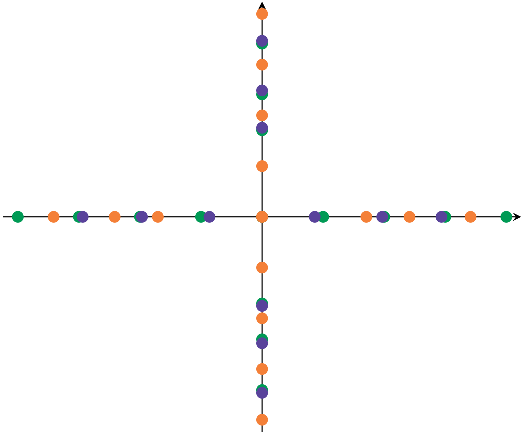
The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s
Orange: roots of t
$\gamma = 0.1$

Roots of $\lambda \mapsto F_1(\lambda)$

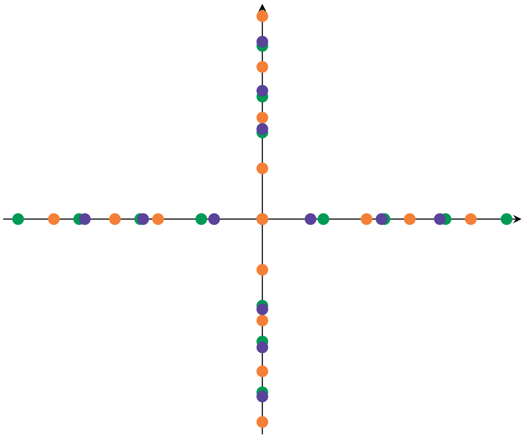
The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s
Orange: roots of t
$\gamma = 0.2$

Roots of $\lambda \mapsto F_1(\lambda)$

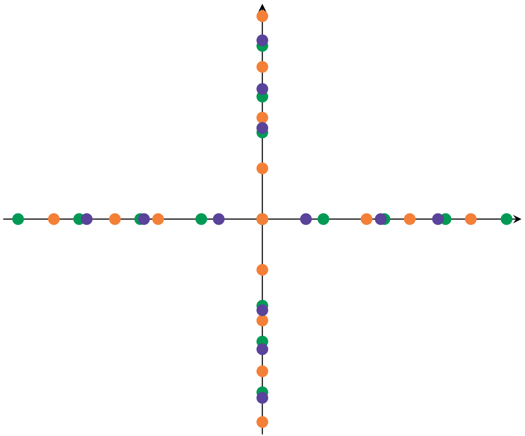
The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s
Orange: roots of t
$\gamma = 0.3$

Roots of $\lambda \mapsto F_1(\lambda)$

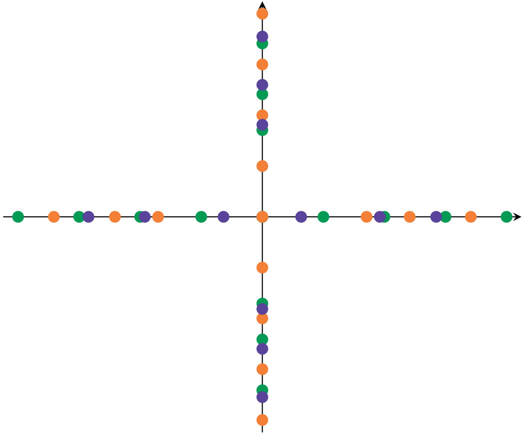
The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s
Orange: roots of t
$\gamma = 0.4$

Roots of $\lambda \mapsto F_1(\lambda)$

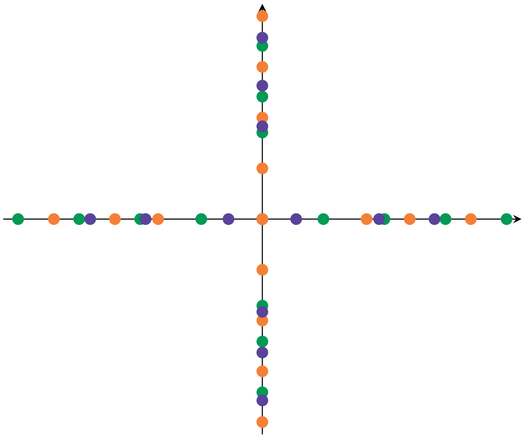
The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s
Orange: roots of t
$\gamma = 0.5$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



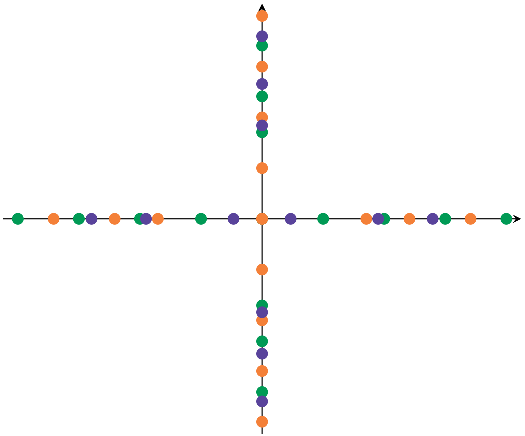
Green: roots of s

Orange: roots of t

$\gamma = 0.6$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



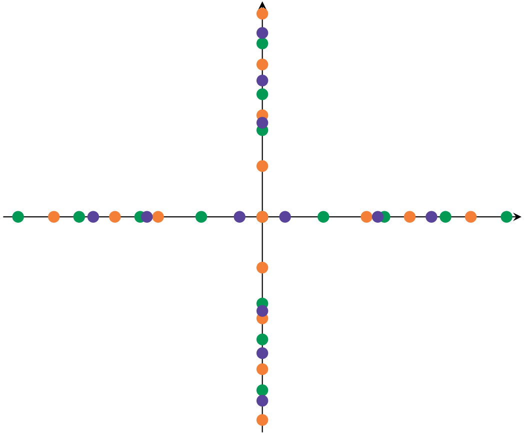
Green: roots of s

Orange: roots of t

$\gamma = 0.7$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



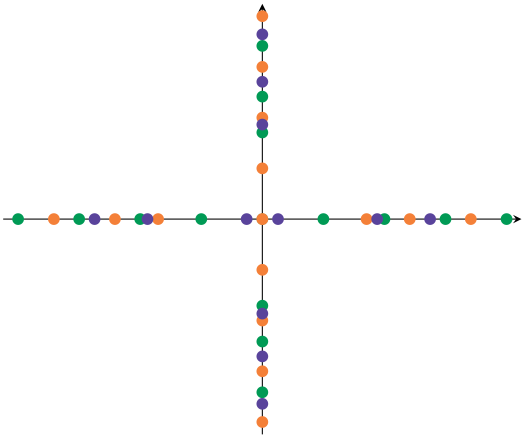
Green: roots of s

Orange: roots of t

$\gamma = 0.8$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



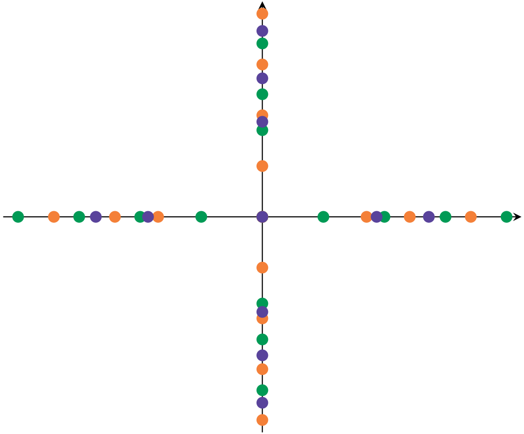
Green: roots of s

Orange: roots of t

$\gamma = 0.9$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s

Orange: roots of t

$\gamma = 1 \rightsquigarrow$ roots of F_1

Roots of $\mu \mapsto F_1(\lambda(\mu))$

Graphical representation in the μ plane

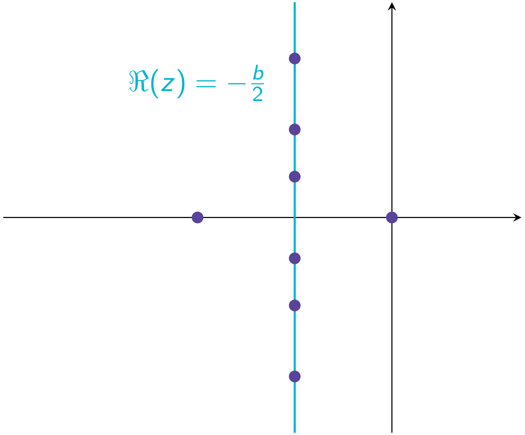
$$\lambda(\mu) = \sqrt[4]{-\frac{b\mu + \mu^2}{a}}$$

i.e.

$$\mu^2 + b\mu + a\lambda^4 = 0$$

Roots of $\mu \mapsto F_1(\lambda(\mu))$

Graphical representation in the μ plane



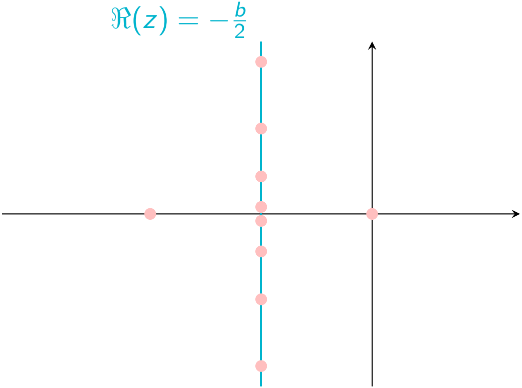
$$\lambda(\mu) = \sqrt[4]{-\frac{b\mu + \mu^2}{a}}$$

i.e.

$$\mu^2 + b\mu + a\lambda^4 = 0$$

Moving α from 0 to $+\infty$

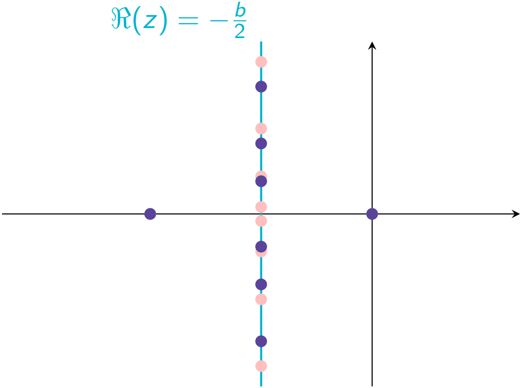
Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



Pink: roots of F_0 ($\alpha = 0$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$

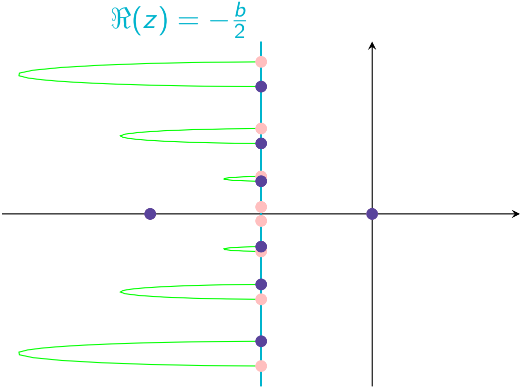


Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$

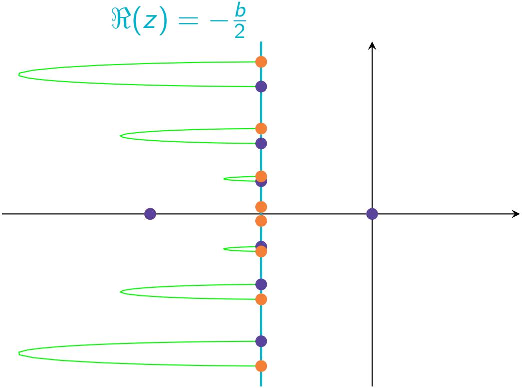


Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



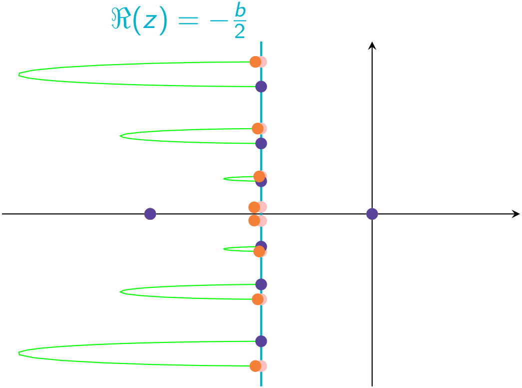
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



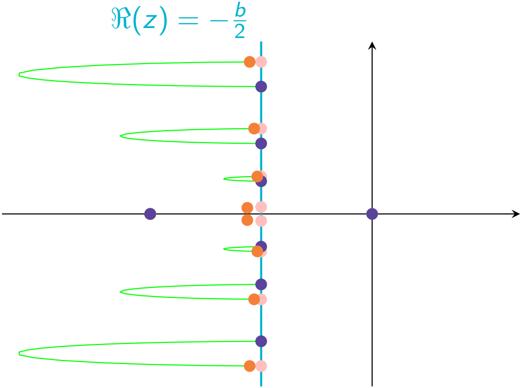
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.1$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



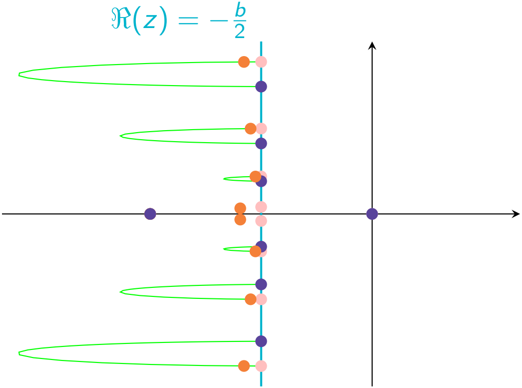
Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 0.2)$

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



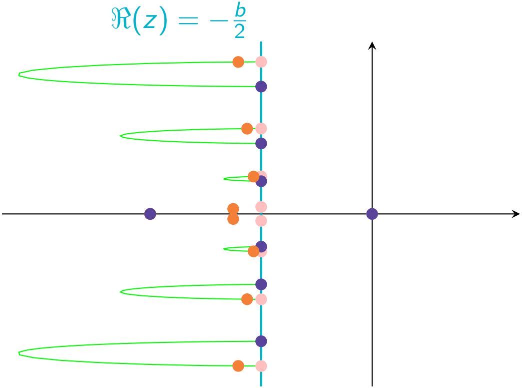
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.3$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



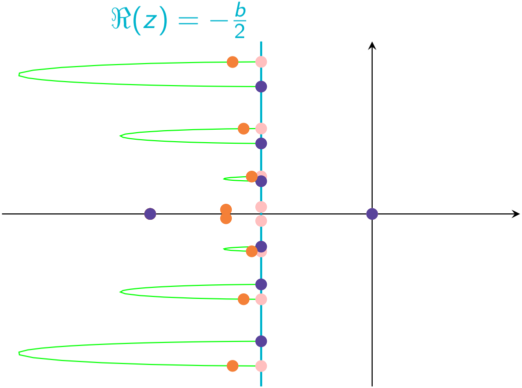
Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 0.4)$

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



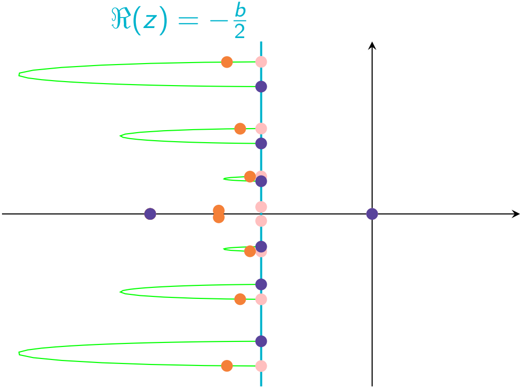
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.5$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



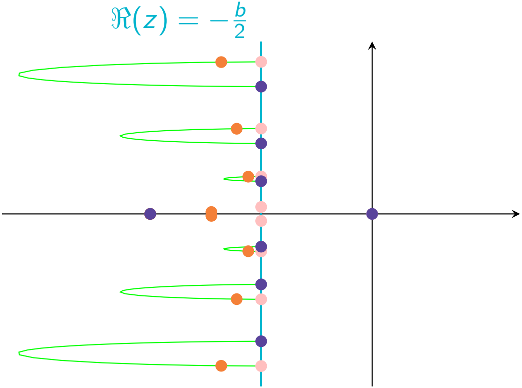
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.6$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



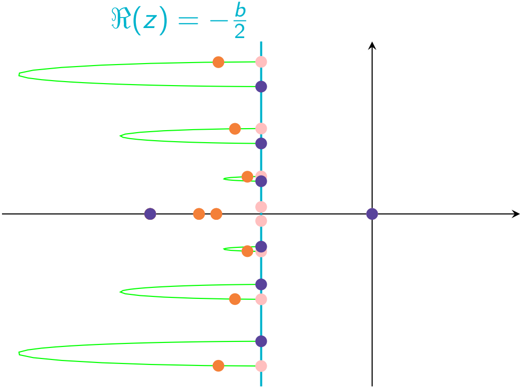
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.7$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



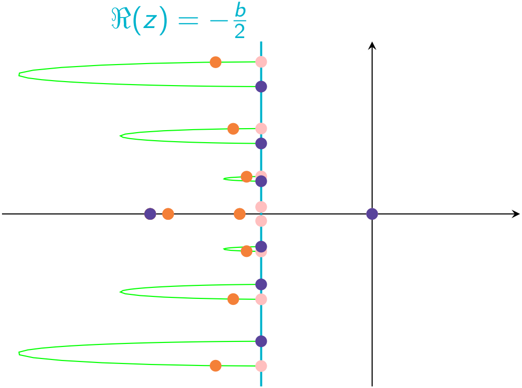
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.75$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



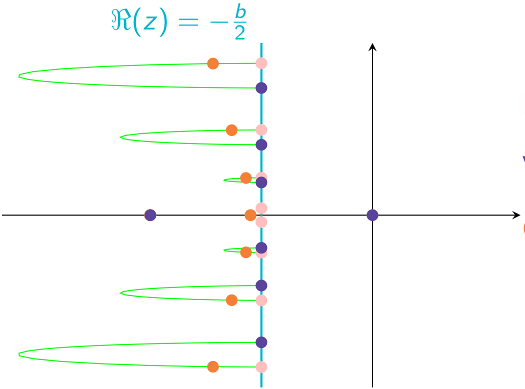
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.8$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



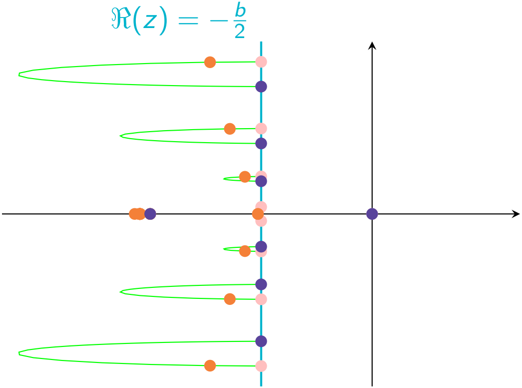
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.85$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



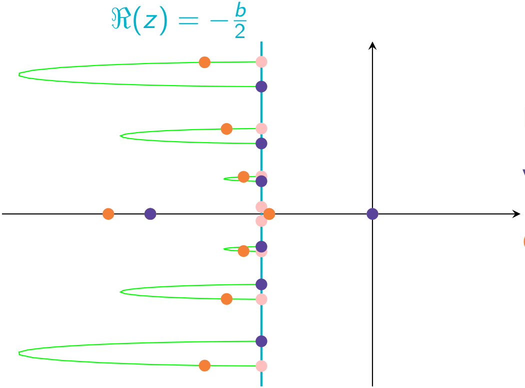
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.9$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



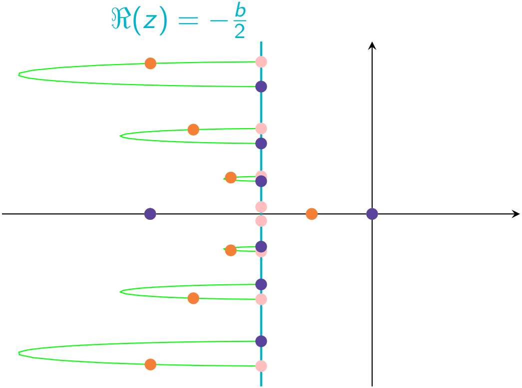
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 1$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



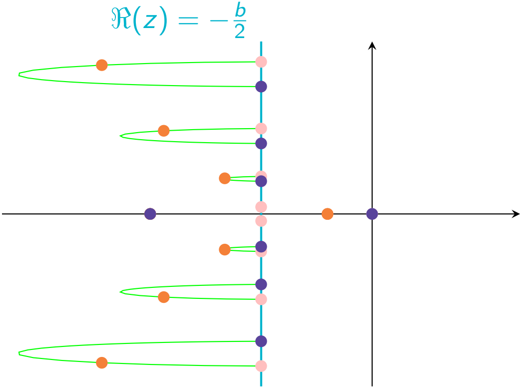
Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 2)$

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



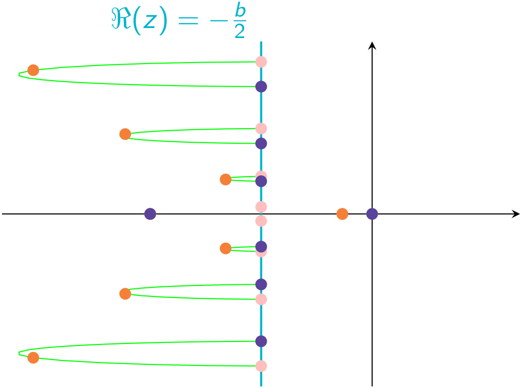
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 3$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



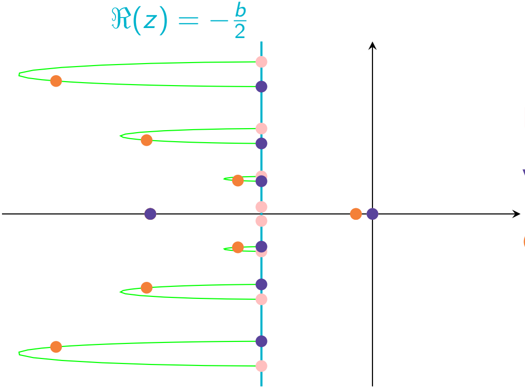
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 5$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



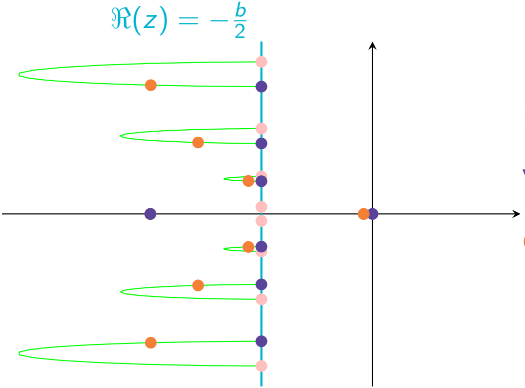
Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 10)$

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



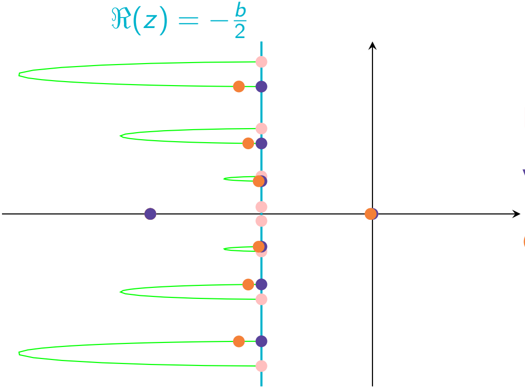
Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 20)$

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



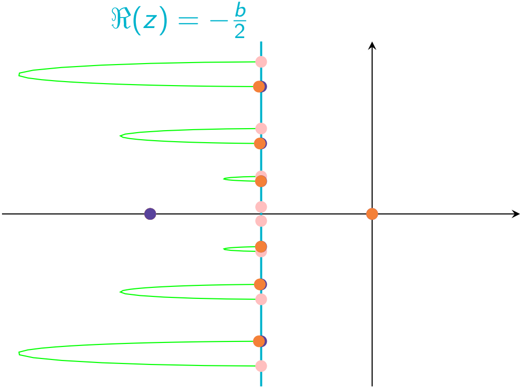
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 100$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 1000$)

Main results

Theorem (G., Régnier, Troestler (2023))

Recall that the optimal decay rate $\omega(\alpha)$ *is given by*

$$\omega(\alpha) = \sup\Big\{\Re(\mu) \mid \mu \text{ is an eigenvalue of the problem}\Big\}.$$

Main results

Theorem (G., Régnier, Troestler (2023))

Recall that the optimal decay rate $\omega(\alpha)$ is given by

$$\omega(\alpha) = \sup\left\{\Re(\mu) \mid \mu \text{ is an eigenvalue of the problem}\right\}.$$

Then,

1. *ω is continuous in α.*

Main results

Theorem (G., Régnier, Troestler (2023))

Recall that the optimal decay rate $\omega(\alpha)$ is given by

$$\omega(\alpha) = \sup\Big\{\Re(\mu) \mid \mu \text{ is an eigenvalue of the problem}\Big\}.$$

Then,

1. ω *is continuous in* α*.*
2. ω *is nondecreasing;*

Main results

Theorem (G., Régnier, Troestler (2023))

Recall that the optimal decay rate $\omega(\alpha)$ is given by

$$\omega(\alpha) = \sup\Big\{\Re(\mu) \mid \mu \text{ is an eigenvalue of the problem}\Big\}.$$

Then,

1. ω *is continuous in* α.
2. ω *is nondecreasing;*
3. *one has*

$$\lim_{\alpha\to+\infty} \omega(\alpha) = 0. \qquad (!)$$

A physical conclusion

The term $\alpha\partial_t u(\xi, t)\delta_\xi$ is **definitely not** a damping term in the problem.

Thanks for your attention!

References

K. Ammari, M. Dimassi, M. Zerzeri. *The rate at which energy decays in a viscously damped hinged Euler-Bernoulli beam*, J. Diff. Equ., vol. 257, issue 9 (2014), 3501–3520.

A-R. Liu, C-H. Liu, J-Y. Fu, Y-L. Pi, Y-H. Huang, J-P. Zhang. *A Method of Reinforcement and Vibration Reduction of Girder Bridges Using Shape Memory Alloy Cables.* Int. J. Struct. Stab. Dyn., vol. 17, No. 7 (2017) 1750076.

V. Régnier. *Do Shape Memory Alloy cables restrain the vibrations of girder bridges? - A mathematical point of view.* ESAIM: COCV, vol. 29, No. 16 (2023), 24 pages.

D. Galant, V. Régnier, C. Troestler. *Do Shape Memory Alloy cables restrain the vibrations of girder bridges? — a mathematical answer. In preparation.*